Chapter 8

Well Temperament

8.1. The Basic Problem with the Tuning of Keyboard Instruments and a Summary of the Different Philosophies of its Solution

Let us imagine the manner in which the idea of our present keyboard has developed. Early on, musicians must have realized that consonant intervals such as octaves, fifths, and thirds could be tuned by listening to beats. For example, imagine an instrument with a number of strings. When we tune a scale by means of fifths, we start, say, with C, and tune one fifth up to G. The next fifth would produce a D more than an octave above the first C. This D would then be brought down an octave. Early musicians must have realized that, rather than tune the *D* in the two steps just described, it would be easier to tune the D directly one fourth down from G. Similarly, we can next tune an A one fifth up from the D, and then tune an E down one fourth from the A. Continuing in this manner, it is possible to tune twelve successive notes to get C, C♯, D, D♯, E, E♯, F♯, G, G♯, A, A♯, B. Upon trying to tune the next note, B♯, it would be noticed that this B♯ is not the same as C but sharp (by 23.5¢). Similarly, it would be noticed that the E♯ is sharper than the F tuned up one perfect fourth from C. Moreover, it would also be disturbing that the interval C—E is not perfect but wider than perfect (by 21.5¢). These imperfections are characteristic of the so-called Pythagorean tuning.

An obvious solution is to try to slightly compress the fifths (and, consequently, widen the fourths) each equally by such an amount that the B♯$_7$, when lowered seven octaves, would be the same as C. This manner of tuning is equal temperament. The small amount by which the fifths are compressed and the fourths are widened is not at all objectionable and possibly even an improvement because of their slow beat. Also, the already wide major third C—E is helped because it is compressed by one-third of the (24.5¢) discrepancy (namely, 23.5/3 = 7.8¢). Whereas this major third is helped by this *tempering*, it is still 21.5 – 7.8 = 13.7¢ wide.

The concept of equal temperament was known but was rejected because the most important consonant interval, the major third, still had an unacceptably high beat rate, as well as all other intervals except the octave and fifth.

The next step was to try to tune slightly more-compressed fifths up and more-expanded fourths down such that the major third C–E would be perfect. The amount by which each of these intervals would be tempered is 1/4 of a syntonic comma; namely, 21.5/4 = 5.4¢. However, if the process of tuning fifths and fourths tempered to this extent is continued, we end up with a B♯, which, instead of being sharper than C by 23.5¢, is off by 23.5 – 3(21.5) = – 23.5 – 64.5 = –41.0¢. That is, the B♯ is 41¢ flatter than a C. This error is almost twice as large as the amount (23.5¢) by which the B♯ would be sharp from C in the Pythagorean tuning. Similarly, the E♯ would be 41¢ flatter than an F, and an A♯ 41¢ flatter than a B♭, with the same rule applying to all chromatic pairs. In this, the meantone temperament, all major thirds are perfect, and the fifths and fourths are tempered about three (actually 2.75) times as much as in equal temperament. Consequently, the meantone fifths and fourths beat more than equally tempered fifths but still not enough to be considered objectionable. The problem with meantone is the enormous difference (41/100 of an equally tempered semitone) between chromatic pairs. This means that, for each of the five black notes per octave of the keyboard, a decision must be made whether to tune it as a sharp or a corresponding flat. Once the decision is made, for example, to tune the black note after C as a C♯, a piece containing a D♭ cannot be played unless you are willing to endure some very sour intervals. It is like having one room in which to store junk. The rest of the house is fine, but that room cannot be used in the normal manner.

At the time of Bach, composers wanted not to be limited in their modulations or their key signatures. They wanted to be able to compose in all keys, without performers having to stop in the middle of a concert and re-tune a harpsichord. Or, in the case of the organ, which cannot be easily re-tuned, composers did not want performers to need to avoid certain registers because of a black note tuned to the wrong one of a chromatic pair. On the other hand, in equal temperament, while all keys could be used, none of the thirds sounded good. The major thirds were all about 13¢ wide, and had a strident quality. This stridency could be tolerated in exotic keys, such as C-sharp major, but not in C-major, where the interval C—E was expected to have a virgin-like purity.

The idea must have occurred to temper the first four fifths to make the third C—E perfect. Thus, 21.5¢ of the total required 23.5¢ of tempering was spent. Continuing in the manner of tuning meantone would cause two more errors of 21.5¢, which adds up to another 43¢ of tempering to be applied. This manner of tuning is the cause of the 41¢ meantone wolf. To avoid the wolf and have the difference between chromatic pairs be negligible, it is necessary to spend the remaining 2¢ somewhere without tempering any more. Thus, the rest of the notes are tuned by perfect fourths and fifths, such that the remaining 2¢ appears in the last fifth of the circle. In this, the Aron-Neidhardt 1/4-syntonic-comma well temperament, it is then be possible to play in all keys without re-tuning. Moreover, each key has its own character, which can be exploited by the composer; namely, the key of C is pure and innocent, with each key one step removed successively more strident. To celebrate this new manner of tuning, Bach wrote twenty-four preludes and fugues—one in each major and minor key. He named this work *The Well-Tempered Clavier*. In it we can observe how he embraced the added dimension of every key having a different characteristic. The first prelude has an exploratory nature commensurate with a great mind savoring both the sweetness of the key of C major and the differences in sound as short excursions are tentatively made to adjacent keys. When this exploration is done, the corresponding fugue begins to delve freely into these now-known areas. The next prelude in C-sharp major is extremely fast, to avoid dwelling on intervals with strident beat rates. Instead, their stridency is utilized to increase the brilliance and frenzied excitement contained in this prelude.

Bach was such a genius that he was always looking for challenges and new dimensions of expression. *The Well Tempered Clavier* is readily perceived to have this dimension when played on a harpsichord tuned in well temperament and listened to with the above understanding. How sad Bach would be to know that most of the performances of his music are played on instruments tuned in equal temperament, devoid of this dimension. Equally sad is the fact that present ears are lacking in any training that would enable this dimension to be appreciated.

How is it that the well temperament method of tuning became lost? We may well ask this question about the totality of the works of Bach himself. For it is well known that the works of Bach fell into disuse after his death, only to be revived much later by Mendelssohn. In fact during the "romantic" period, there was a contempt for prior music. In France, harpsichords were burned as firewood. This is not the first time in history that great works were scorned or lost. In ancient China the first emperor, Shi Huangdi, who reigned from 221–207 B.C., ordered all books burned. The fires smoldered for months afterward. In addition, 460 scholars were executed.

8.2. The 1/4-Syntonic-Comma Well Temperament

There are quite a number of different versions of well temperament.[19] We list here a typical one, the Aron-Neidhardt 1/4-syntonic-comma well temperament. The tuning scheme for this temperament can be represented in comma notation as follows:

[19] See Owen Jorgensen, *Tuning the Historical Temperaments by Ear,* The Northern Michigan University Press, Marquette, MI, 1977, p. 309. This book is "a manual of eighty-nine methods for tuning fifty-one scales on the harpsichord, piano, and other keyboard instruments" and contains a wealth of information on the various well temperaments as well as other historical tuning systems.

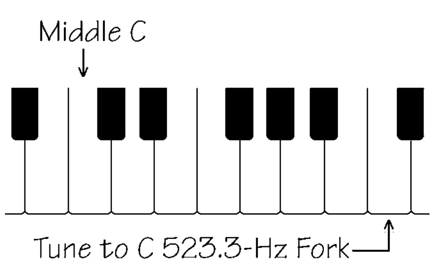
$$\overbrace{C\flat^0 \leftarrow G\flat^0 \leftarrow D\flat^0 \leftarrow A\flat^0 \leftarrow E\flat^0 \leftarrow B\flat^0 \leftarrow F^0 \leftarrow C^0}^{\text{Seven perfect (Pythagorean) fifths}} \overbrace{\rightarrow G^{-1/4\,S} \rightarrow D^{-1/2\,S} \rightarrow A^{-3/4\,S} \rightarrow E^{-1\,S}}^{\text{C—E is a perfect third}}$$

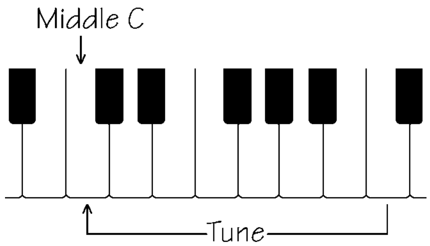
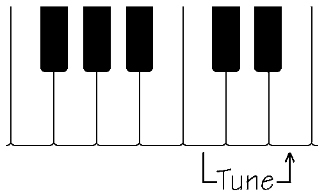
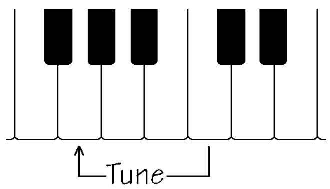
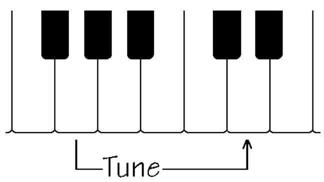
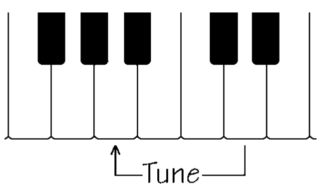
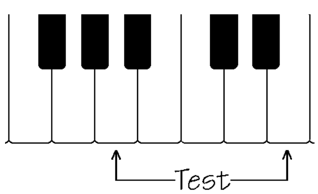
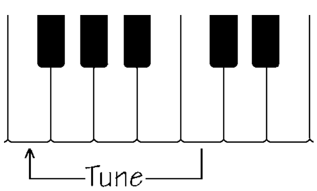
This representation illustrates that, starting on C, we tune down perfect fifths to successively tune the notes F, B♭, E♭, A♭, D♭, G♭, and C♭ (B will be enharmonic with C♭). Then we tune up four fifths, each tempered by reducing it by 1/4 of a syntonic comma. This procedure results in C—E being a perfect third when the tuned E is then raised an appropriate number of octaves. The syntonic comma is now distributed evenly between the four fifths C—G, G—D, and D—A, and A—E. There will be only one perfect third; namely, C—E. However, it also is the case that if we tuned a new note one perfect fifth up from E^{-1S}, we would produce B^{-1S}. Let us ask how this B differs from the $C\flat^0$. Because the C♭ is tuned solely by perfect fifths, it is one ditonic comma less than a B^0 tuned upward from C by perfect fifths. However, B^{-1S} is one syntonic comma less than B^0. Thus $C\flat^0$ is lower than B^{-1S} by $-1S - (-1D) = 1D - 1S = 23.46001 - 21.50629 = 1.95372$¢. (This difference between a ditonic and syntonic comma is called a schisma and is, by coincidence, very close to the amount 1.955002¢ by which equally tempered fifths are narrowed.) Similarly, G♭ differs by a schisma from the F♯ that would be tuned one perfect fifth up from B^0, etc. Therefore, this tuning would be considered *enharmonic*; that is, the one-schisma difference between chromatic pairs is small enough to be neglected, and a black note can be used for both its corresponding sharp and flat.

Because the tempering uniformly distributes the syntonic comma among four fifths, this tuning would be characterized as a *1/4-syntonic-comma temperament.* In fact the name for this tuning is *1/4-syntonic-comma well temperament*. Because this tuning uses fifths tempered by differing amounts, it would be called *irregular,* and because the wolf is eliminated, it is called *closed.*

8.3. Tuning the Aron-Neidhardt 1/4-Syntonic-Comma Well Temperament

This temperament is extremely easy to tune. First tune C, E, G, D, and A of the temperament octave exactly as in tuning meantone. Then tune F_3, a perfect fifth down from C_4, $B\flat_3$ a perfect fourth up from F_3, $E\flat_4$ a perfect fourth up from $B\flat_3$, $A\flat_3$ a perfect fifth down from $E\flat_4$, $D\flat_4$ a perfect fourth up from $B\flat_3$, $G\flat_3$ a perfect fifth down from $D\flat_4$, and $C\flat_3$ a perfect fourth up from $G\flat_3$. The $C\flat_3$ just tuned should form a slightly widened fourth when the interval $C\flat_3$—E_4 is played. That is, $C\flat_3$ differs from B_3 by about 2¢. This difference is the schisma. Similarly all such pairs will differ by only a schisma and can be considered to been enharmonic to within one schisma, which is essentially negligible. Therefore this temperament can be played in all keys without re-tuning. For the beat rates in the following scheme, the superscripts *n* and *w* stand for *narrow* and *wide*, respectively.

A Scheme for Tuning the Temperament Octave in the 1/4-Syntonic-Comma Aron-Neidhardt Well Temperament in the Key of C			
Step	**View**	**Beat Rate (Hz)**	**Comments**
1		zero	Tune the C above middle C to a 523.3-Hz tuning fork.

2		1.6^{n}	Tune middle C to the C above middle C. This note is tuned sharp by 1.6 b/s to achieve the A440-Hz standard.
3		zero	Tune the E above middle C to middle C so that C—E is a perfect (beatless) third.
4		2.4^{w}	Tune the G below middle C to middle C so that G—C is a fourth, widened by 2.4 b/s.
5		1.8^{n}	Tune the D above middle C to the G below middle C so that G—D is a fifth, narrowed by 1.8 b/s.
6		2.7^{w}	Tune the A below middle C to the D above middle C so that A—D is a fourth, widened by 2.7 b/s.
7		2.0^{n}	Test the fifth A—E. It should be 2.0 b/s narrow. If not, adjust the tuning in steps 4–6 above.
8		zero	Tune the F below middle C to middle C so that F—C is a perfect (beatless) fifth.

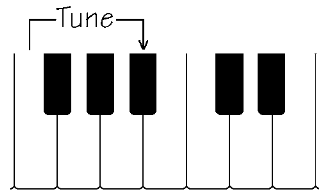
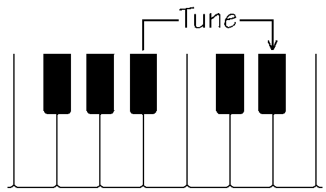
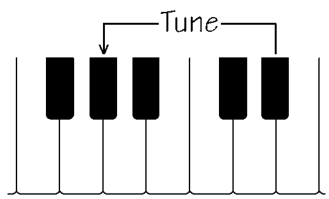
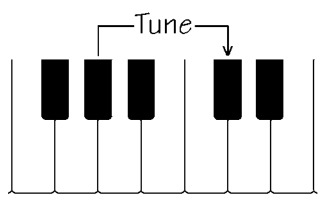
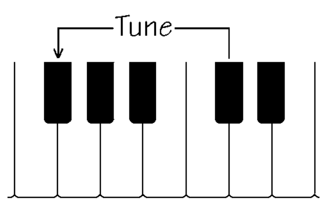
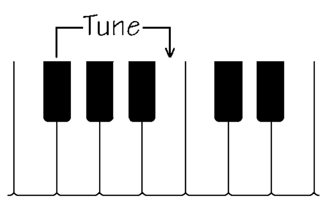
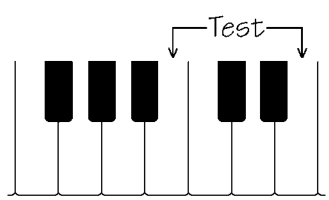
9		zero	Tune the B♭ below middle C to the F below middle C so that F—B♭ is a perfect (beatless) fourth.
10		zero	Tune the E♭ above middle C to the B♭ below middle C so that B♭—E♭ is a perfect (beatless) fourth.
11		zero	Tune the A♭ above middle C to the E♭ above middle C so that A♭—E♭ is a perfect (beatless) fifth.
12		zero	Tune the D♭ above middle C to the A♭ below middle C so that A♭—D♭ is a perfect (beatless) fourth.
13		zero	Tune the G♭ below middle C to the D♭ above middle C so that G♭—D♭ is a perfect (beatless) fifth.
14		zero	Tune the C♭ below middle C to the G♭ below middle C so that G♭—D♭ is a perfect (beatless) fourth.
15		1.1[w]	Test the interval C♭—E. It should be a widened "fourth," beating at 1.1 Hz. If not, repeat steps 8–14. Note that this interval would be a "wolf" in meantone but is no problem here.

Fig. 8.1. *The procedure for tuning the twelve notes of the temperament octave in the 1/4-syntonic-comma Aron-Neidhardt well temperament.*

Note that tuning the above temperament is a bit harder than tuning meantone because any error in tuning the fifths is cumulative, and correcting an error requires revisiting the whole sequence of perfect fifths. On the other hand, tuning the perfect intervals that occur in both meantone and the above well temperament is much easier than tuning the intervals of equal temperament, all of which are tempered. Moreover, the tuning of equal temperament is much more critical and exacting because the beat rates of the fifths are so slow that errors are hard to hear. Precise tuning requires the use of numerous tests.

8.4. A Temperament Used by Bach: 1/5-Pythagorean-Comma Well Temperament

Herbert Anton Kellner attributes the following system of tuning to Andreas Werckmeister in 1691. Kellner claims that Bach used this tuning for the Well Tempered Clavier and called it *Wohltemperirt*.[20] It is somewhat harder to tune than the 1/4-comma well temperament but sounds correspondingly better in keys remote from C major.

$$G\flat^{0} \leftarrow D\flat^{0} \leftarrow A\flat^{0} \leftarrow E\flat^{0} \leftarrow B\flat^{0} \leftarrow F^{0} \leftarrow C^{0} \rightarrow G^{-1/5\,P} \rightarrow D^{-2/5\,P} \rightarrow A^{-3/5\,P} \rightarrow E^{-4/5P} \rightarrow B^{-4/5P}.$$

The numerical results are listed in Table 8.1.

Note	Fractional Ratio to C	Decimal Ratio to C	Frequency (Hz), Based on A440	Cents from E. T.[21]
F_3	$(2/3)^1$	0.666666667	175.246244331	+6.25
$G\flat_3$	$(2/3)^6\ 2^2$	0.702331962	184.621557829	–3.52
G_3	$(3/2)^1\ P^{-1/5}/2$	0.747970097	196.618425458	+5.47
$A\flat_3$	$(2/3)^4\ 2^2$	0.790123457	207.699252491	+0.39
A_3	$(3/2)^3\ P^{-3/5}/2^2$	0.836917603	220.000000000	0.00
$B\flat_3$	$(2/3)^2\ 2$	0.888888889	233.661659020	+4.30
B_3	$(3/2)^5\ P^{-4/5}/2^3$	0.938984011	246.830131998	–0.78
C_4	1	1.000000000	262.869366365	+8.21
$D\flat_4$	$(2/3)^5\ 2^3$	1.053497942	276.932336480	–1.57
D_4	$(3/2)^2\ P^{-2/5}/2^1$	1.118918532	294.129405520	+2.74
$E\flat_4$	$(2/3)^3\ 2^2$	1.185185185	311.548878606	+2.34
E_4	$(3/2)^4\ P^{-4/5}/2^2$	1.251978681	329.106842576	–2.73

Table 8.1. *Frequencies of the temperament octave of the 1/5-Pythagorean-Comma Well Temperament scale founded on middle C = 262.9 Hz.*

Note that in this tuning, E is sharp from being a perfect third with C by $S - 4/5\,P$ = 21.5 cents – 4(23.5)/5 cents = 2.7¢, which is small compared to the amount by which E is sharp from being a perfect third with C in equal temperament, namely, $S - 4/12\,P$ = 21.5 cents – 4(23.5)/12 cents = 13.6¢.

To tune Wohltemperirt, proceed in a similar manner to that of tuning 1/4-comma well temperament except that the beat rates of the first four fifths are different here (see Table 8.2 below).

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[20]See http://ha.kellner.bei.t-online.de/#spezifikation.

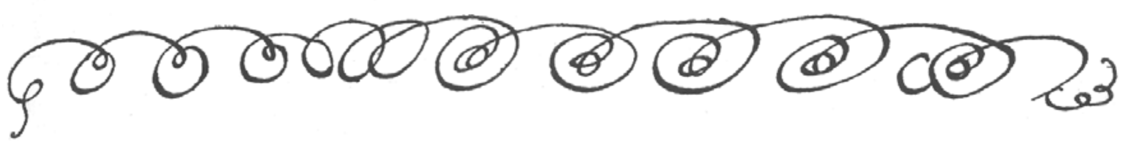
[21]This column is given for those who want to tune electronically, based on A440

Directions	Interval	Description	Beats/s
tune	G_3—C_4	fourth	2.1 w
tune	G_3—D_4	fifth	1.6 n
tune	A_3—D_4	fourth	2.4 w
tune	A_3—E_4	fifth	1.8 n
test	C_4—E_4	major third	2.1 w
test	F_3—A_3	major third	3.8 w
test	G_3—B_3	"major third"	4.2 w
test	D_3—$G\flat_3$	"major third"	6.3 w
test	A_3—$D\flat_4$	"major third"	7.7 w
test	$G\flat_3$—D_4	"major third"	6.3 w

Table 8.2. *Frequencies of the temperament octave of the 1/5-Pythagorean-Comma Well Temperament scale founded on middle C = 262.9 Hz.*

8.5. The System of Well Temperament Likely Used in Bach's Well-Tempered Clavier

Below is an inscription that Bach drew on the top of the title page of his *Das Wohltemperirte Clavier*. The meaning of this inscription has been a puzzle until very recently.



Bradley Lehman[22] has convincingly interpreted this inscription as representing the tuning sequence Bach used for the Well Tempered Clavier. When the inscription is viewed upside down, it becomes:

Lehman has ingeniously deciphered the single loops as representing untempered fifths, the double loops as equally tempered fifths, and the triple loops as fifths tempered twice that of equal temperament. The hand-written letter *C*, which is now upside down, shows where C belongs in the tuning sequence. Thus the sequence is:

$F^{+2/12\ P} \leftarrow C^0 \rightarrow G^{-2/12\ P} \rightarrow D^{-4/12\ P} \rightarrow A^{-6/12\ P} \rightarrow E^{-8/12\ P} \rightarrow B^{-8/12\ P} \rightarrow F\sharp^{-8/12\ P} \rightarrow C\sharp^{-8/12\ P} \rightarrow G\sharp^{-9/12\ P} \rightarrow D\sharp^{-10/12\ P} \rightarrow A\sharp^{-11/12\ P}$.

This tuning is even more usable than 1/5-comma well temperament in keys remote from C, which adds credibility to the assertion that Bach used it for his Well Tempered Clavier, which uses all keys in practice—not just theory. The numerical results are listed in Table 8.3, below.

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[22]See Bradley Lehrman, "Bach's Extraordinary Temperament: Our Rosetta Stone," *Early Music*, February, 2005, and http://www.larips.com.

Note	Fractional Ratio to C	Decimal Ratio to C	Frequency (Hz), Based on A440	Cents from E. T.	Cents from E. T.[23]
F_3	$(2/3)^1\ P^{+2/12}$	0.668174039	175.404633854	+1.95	7.82
$F♯_3$	$(3/2)^6\ P^{-8/12}/2^4$	0.705511579	185.206238152	–3.91	1.96
G_3	$(3/2)^1\ P^{-2/12}/2^1$	0.748308032	196.440880223	–1.95	3.92
$G♯_3$	$(3/2)^8\ P^{-9/12}/2^5$	0.792804743	208.121862788	–1.95	3.92
A_3	$(3/2)^3\ P^{-6/12}/2^2$	0.838052481	220.000000000	–5.87	0.00
$A♯_3$	$(3/2)^{10}\ P^{-11/12}/2^6$	0.889893236	233.608892472	–1.95	3.92
B_3	$(3/2)^5\ P^{-8/12}/2^3$	0.940682106	246.941650914	–5.87	-5.87
C_4	1	1.000000000	262.513392643	0.00	5.87
$C♯_4$	$(3/2)^7\ P^{-8/12}/2^4$	1.058267369	277.809357360	–1.95	3.92
D_4	$(3/2)^2\ P^{-4/12}/2^1$	1.119929822	293.996156549	–3.91	1.96
$D♯_4$	$(3/2)^9\ P^{-10/12}/2^5$	1.187864957	311.830459864	–1.95	3.92
E_4	$(3/2)^4\ P^{-8/12}/2^2$	1.254242807	329.255534464	–7.82	-1.95

Table 8.3. *The frequency relationships of Bach's well temperament. Note the very minor deviations from equal temperament (last column). The letter P in the second column stands for the Pythagorean comma, namely, P = 531441/524288 = 1.01364326477.*

For the tuning of the five successive fifths F—C, C—G, G—D, D—A, and A—E, Bach doubled the 1/12-comma tempering of equal temperament, thus making the major third C—E become about 8¢ closer to perfect. More precisely, it reduces by 7.8¢ the 13.7¢ error left by equal temperament, leaving C—E off from perfect by only 5.9¢, which results in a slow-beating third (4.5 b/s). The same benefit is achieved for the interval F—A.

Tuning the first five fifths accumulates about 10¢ of tempering beyond that of equal temperament. Having improved the major thirds in the keys of F and C, Bach let the next three fifths E—B, B—F♯, and —F♯—C♯ go untempered, which brings B, F♯, and C♯ 2, 4, and 6¢, respectively, closer to what they would be if tuned using *1/12*-comma-equal-temperament fifths. The lack of any tempering of these three fifths causes the major thirds of successive keys of G, D, and A to become progressively less pure (increasingly faster beating) than that in the keys of F and C.

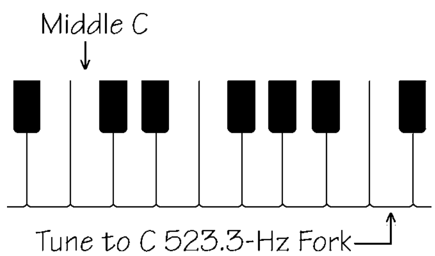
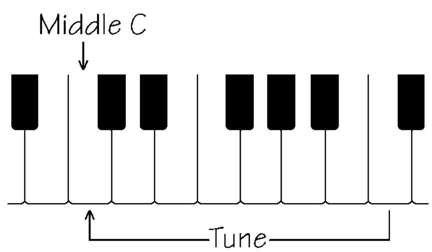
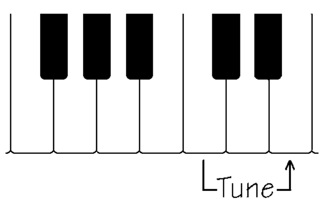
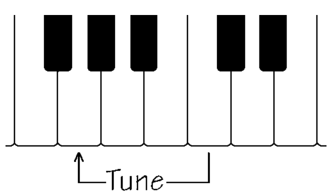
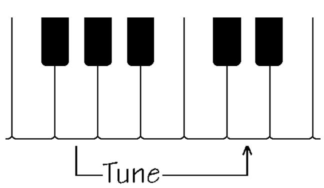
The last three fifths C♯—G♯, G♯—D♯, and D♯—A♯ are tempered the same amount as in equal temperament. The result is that 4¢ of extra tempering (twice that of equal temperament) is hidden in the last interval A♯—F. The extra 4¢ simply floats at the end (like the curlicues at each end of Bach's series of loops). The result is that the interval F—A♯, which doubles as the fourth F—B♭, is actually narrowed instead of widened as in equal temperament.

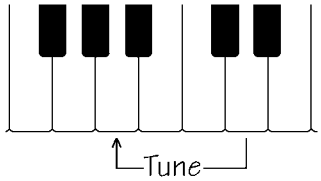
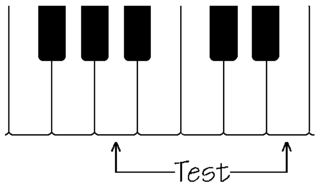
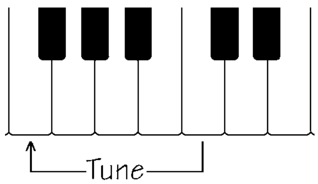
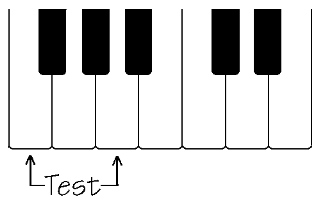
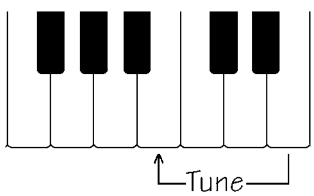
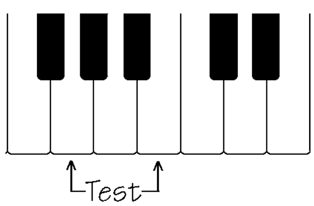
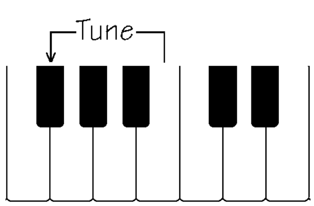
The reader should verify the quantities in Table 8.3.

[23]This column is given for those who want to tune electronically, based on A440.

8.6. Tuning the Well Temperament Likely Used in Bach's Das Wohltemperirte Clavier

For the beat rates in the following scheme, the superscripts *n* and *w* stand for *narrow* and *wide*, respectively.

A Scheme for Tuning the Temperament Octave of a Well Temperament Likely Used in Bach's Das Wohltemperirte Clavier			
Step	**View**	**Beat Rate (Hz)**	**Comments**
1		0	Tune the C above middle C to a 523.3-Hz tuning fork.
2		2^n	Tune middle C to the C above middle C. This note is tuned sharp by 2 b/s to achieve the A440-Hz standard.
3		4.5^w	Tune the E above middle C to middle C so that C—E is a third, widened by 4.5 b/s.
4		1.8^w	Tune the G below middle C to middle C so that G—C is a fourth, widened by 1.8 b/s.
5		1.3^n	Tune the D above middle C to the G below middle C so that G—D is a fifth, narrowed by 1.3 b/s.

6		2.0^{w}	Tune the A below middle C to the D above middle C so that A—D is a fourth, widened by 2.0 b/s.
7		1.5^{n}	Test the fifth A—E. It should be 1.5 b/s narrow. If not, adjust the tuning in steps 4–6 above.
8		1.2^{n}	Tune the F below middle C to middle C so that F—C is a fifth, narrowed by 1.2 b/s.
9		3.0^{w}	Test the third F—A. It should be 3.0 b/s wide. If not, adjust the tuning in steps 4–6 and 8 above.
10		zero	Tune the B below middle C to the E above middle C so that B—E is a perfect (beatless) fourth.
11		5.6^{w}	Test the third G—B. It should be 5.6 b/s wide. If not, adjust the tuning in steps 4–6 and 10 above.
12		zero	Tune the F♯ below middle C to the B below middle C so that F♯—B is a perfect (beatless) fourth.

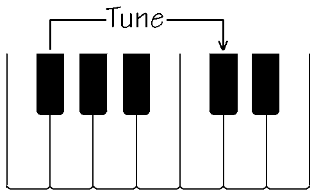
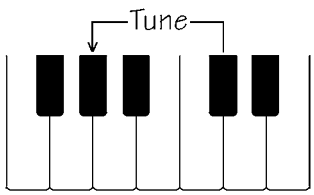
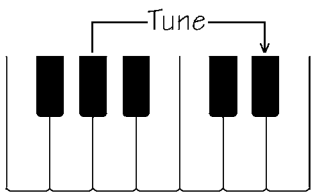
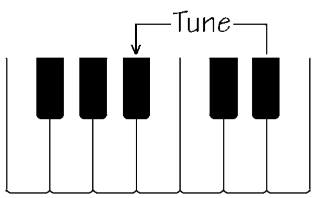
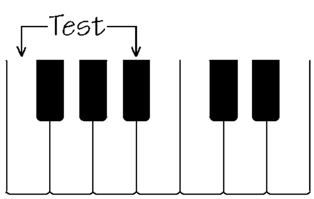
13		zero	Tune the C♯ above middle C to the F♯ below middle C so that F♯—C♯ is a perfect (beatless) fifth.
14		0.9^{w}	Tune the G♯ below middle C to the C♯ above middle C so that G♯—C♯ is a fourth, widened by 0.9 b/s.
15		0.7^{n}	Tune the D♯ above middle C to G♯ so that G♯—D♯ is a fifth, narrowed by 0.7 b/s.
16		1.1^{w}	Tune the A♯ below middle C to the D♯ above middle C so that A♯—D♯ is a fourth, widened by 1.1 b/s.
17		0.8^{n}	Test the interval F—A♯. It should be a narrowed[24] "fourth," beating at 0.8 Hz. If not, repeat steps 11–16.

Fig. 8.2. *The procedure for tuning the twelve notes of the temperament octave in Bach's well temperament.*

8.7. Other Well Temperaments

There are over a dozen thoroughly documented well temperaments.[25] The 1/2-Erlangen-comma well temperament (Fig. 8.2) is one from which the reader can learn much and become self-sufficient by attempting to work out the ratios, frequencies, and beat rates (Ex. 8-1). The reader should then attempt to tune this temperament.

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[24]Because of the sharpening of F by 1/12 comma more than in equal temperament and of the lowering of A♯, by 1/12 of a comma more than in equal temperament, the fourth F—B♭', played by using the interval F—A♯, actually is 1/12 of a comma narrow instead of 1/12 comma wide, as in equal temperament.

[25]See Jorgensen, Owen, *Tuning the Historical Temperaments by Ear*, Northern Michigan University Press, Marquette, MI, 1977, pp. 245–342.

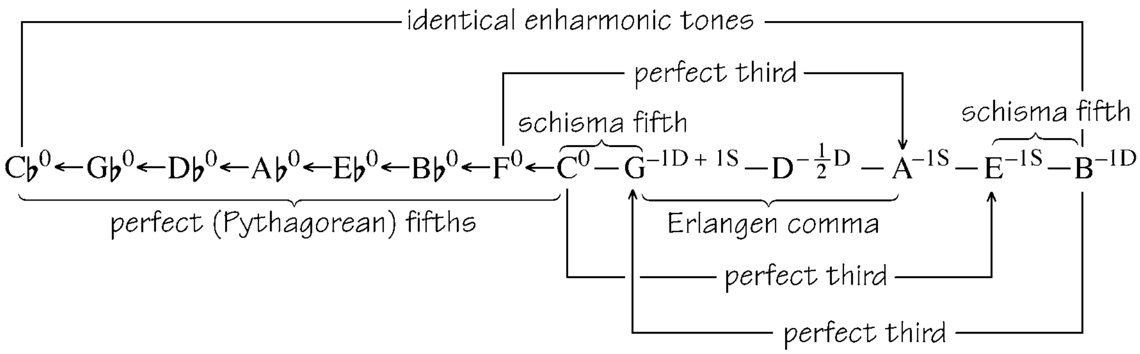

Fig. 8.2. *The 1/2-Erlangen-Comma Well Temperament*

Perfect fifths are tuned down from C until C♭ is reached (B will be enharmonic with C♭, thus the designation B^{-1D}). Then A is tuned a perfect third with F (thus, the designation A^{-1S}), and E is tuned a perfect third with C (thus, the designation E^{-1S}).

The syntonic comma must be somehow distributed between the three fifths C—G, G—D, and D–A. However, it is decided, evidently for the sake of a certain symmetry about C, that G is tuned such that G—B is a perfect third (thus the designation G^{-1D+1S}). The interval G—B is technically A♭♭—C♭, but these two intervals differ by only a schisma. Note that the fifth C^0—G^{-1D+1S} is narrowed by a schisma. The remaining narrowing, which is termed the *Erlangen comma E*, is therefore

$$E = 1 \text{ syntonic comma} - 1 \text{ schisma.}$$

$$= 1S - 1s = 1S - (1D - 1S)$$

$$= 2S - 1D = 2(21.50628960) - 23.46001077$$

$$E = 19.55256843¢$$

The Erlangen comma is equally divided between the two fifths G–D and D–A. Therefore,

$$\tfrac{1}{2}E = S - \tfrac{1}{2}D.$$

The 1/2-Erlangen-comma well temperament is easy to tune because all of the notes other than D are tuned using perfect intervals. In practice, it is better to divide the Erlangen comma slightly unequally: D_4 is lowered from a perfect fourth with G_4 until the beat rate of D_4–G_4 is the same as that of the major third B♭$_3$—D_4.[26] The Erlangen comma can be alternatively expressed as follows:

Seven fifths up from C♭ to C, one third and two octaves up from C to E, one fifth down from E to A, two fifths down, compressed by one Erlangen comma from A to G, and one third and two octaves up from G to B.

We end up seven octaves higher than we started. Thus,

$$7F + 1T + 2O - 1F - 2(F - E) + 1T - 2O = 7O.$$

[26]Owen Jorgenson, *Tuning the Historical Temperaments by Ear*, Northern Michigan University Press, Marquette, MI, 1977, pp. 265–272.

Therefore,

$$E = 3O - 4F - 2T = 2^3/(3/2)^4(5/4)^2 = 2048/2025$$

$$= 1.011358025$$

$$E = 19.55256934¢$$

Exercises

8-1 Calculate the frequencies of the 1/2-Erlangen-comma well temperament from C_4–B_4, based on C_4 = 261.626 Hz. Using these frequencies, calculate the beat rates of all of the major thirds and fifths.

8.8. In Closing. The advent of pianos marked the beginning of a dark period in tuning. Harpsichords and harpsichord music were discarded along with the various newly invented closed tunings. Moreover, with the greater difficulty of tuning pianos, tuning became a specialized profession, divorced from the musicians who play them. With the recent revelation that J. S. Bach had in mind one of the various well temperaments when he wrote his *Das wohltemperirte Klavier,* knowledgeable musicologists now understand that equal temperament is *not* the only solution for playing in all keys without re-tuning.

Unfortunately, from the early to mid-twentieth century, musicologists, theoreticians, and piano tuners were quite unable to think in any terms other than equal temperament. For example, an otherwise excellent treatise on piano tuning written at that time by William Braid White[27] contains a treatment of meantone temperament along with instructions for tuning it. The method shown involves tuning the temperament octave in meantone—including the perfect thirds—solely by using tempered fifths! There is no mention of even testing the smoothness of the thirds. The table showing the frequencies of the meantone temperament fails to distinguish between what, in equal temperament, would be enharmonic pairs but which, in meantone, all differ by 41¢ (about 2/5 of a semitone!). White then complains that, “the deplorable effects of having one tone for both a Sharp and a Flat … are plainly to be seen.” Of course, music written in this temperament was never designed to have a sharp substituted for a flat. That music was written and played in such way that a substitution would be unnecessary and, except for one or two cases, would be considered quite inappropriate.

Additionally, one need only look up *The Well-Tempered Clavier* in all but the most recent dictionaries of music to find the erroneous assertion that that name refers to “the then novel system of equal temperament.”[28]

The characterization of J.S. Bach’s *Das wohltemperirte Clavier* as *The Well Tempered Clavichord* is another prevalent error. Whereas, in German, *Clavier* refers to the clavichord, it also refers to a keyboard instrument. *Das wohltemperirte Clavier,* refers to the well-temperament tuning of keyboard instruments in general—not a clavichord tuned in equal temperament.

Today, the mathematical relationships of the numerous well temperaments have been thoroughly researched, established, and documented by music historians in books and journals. So has the fact that J. S. Bach wrote music intended to be played in well temperament. It is my sincere hope that as many people as possible will understand Bach’s intention and appreciate some of the dimensions of Bach’s music now disregarded by many.

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[27] White, William Braid, *Piano Tuning and Allied Arts,* Tuner’s Supply Company, Boston, MA, 1945, pp. 236–44.

[28] Apel, Willi, Harvard Dictionary of Music, The Belknap Press of Harvad University Press, Cambridge, MA, 1969, p. 924.